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# Multiple zero modes of the Dirac operator in three dimensions

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**Abstract**

One of the key properties of Dirac operators is the possibility of a degeneracy of zero modes. For the Abelian Dirac operator in three dimensions the construction of multiple zero modes has been sucessfully carried out only very recently. Here we generalise these results by discussing a much wider class of Dirac operators together with their zero modes. Further we show that those Dirac operators that do admit zero modes may be related to Hopf maps, where the Hopf index is related to the number of zero modes in a simple way.

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# 1 Introduction

Fermionic zero modes of the Dirac operator $D_A = \gamma^\mu(\partial_\mu - iA_\mu)$ are of importance in many places in quantum field theory and mathematical physics [1, 2, 3]. They are the ingredients for the computation of the index of the Dirac operator and play a key rôle in understanding anomalies. In Abelian gauge theories, which is what we are concerned with here, they affect crucially the behaviour of the Fermion determinant $\det(D_A)$ in quantum electrodynamics. The nature of the QED functional integral depends strongly on the degeneracy of the bound zero modes.

In three dimensions – which is the case which we want to study here – the first examples of such zero energy Fermion bound states were obtained only in 1986 [4], and some further results have been found recently [5]. In both articles no degeneracy of these zero modes has been observed, because, by their very methods, the authors of [4] and of [5] could only construct one zero mode per gauge field. Only very recently we were able to give the first examples of Dirac operators that admit more than one zero mode [6], thereby establishing that the phenomenon of zero mode degeneracy exists for the Abelian Dirac operator in three dimensions. It is the purpose of this article to generalise and further explain the results of [6].

It should be emphasised here that the problem of the existence and degeneracy of zero modes of the Abelian Dirac operator in three dimensions, in addition to being interesting in its own right, has some deep physical implications. The authors of [4] were mainly interested in these zero modes because in an accompanying paper [7] it was proven that one-electron atoms with sufficiently high nuclear charge in an external magnetic field are unstable if such zero modes of the Dirac operator exist.

Further, there is an intimate connection between the existence and number of zero modes of the Dirac operator for strong magnetic fields on the one hand, and the non-perturbative behaviour of the three dimensional Fermionic determinant (for massive Fermions) in strong external magnetic fields on the other hand. The behaviour of this determinant, in turn, is related to the paramagnetism of charged Fermions, see [8, 9]. So, a thorough understanding of the zero modes of the Dirac operator is of utmost importance for the understanding of some deep physical problems as well.

In addition, it is speculated in [9] that the existence and degeneracy of zero modes for $QED_3$ may have a topological origin as it does in $QED_2$ [10]–[14] — cf. [9] for details and an account of the situation for $QED_{2,3,4}$. We will find some further strong support for that conjecture in our paper.

This article is organised as follows. In Section 2 we briefly review the case of zero modes of the Abelian Dirac operator in two dimensions, because there exists some similarity between the general two-dimensional case and the specific class of zero modes in three dimensions that we want to discuss. We point out some specific features of the two-dimensional case that we shall need later on. In Section 3 we review the features of maps $S^2 \to S^2$ and of Hopf maps $S^3 \to S^2$, because we shall need them for a topological interpretation of our results. In Section 4 we construct our class of Dirac operators together with their zero modes. Further we show that the corresponding magnetic fields may be

related to Hopf maps (they may be expressed as Hopf curvatures of some Hopf maps), and that the Hopf index is related to the number of zero modes of a given Dirac operator in a simple fashion. This topological interpretation of the magnetic fields requires the introduction of a fixed, universal background magnetic field. In the final section we briefly describe another class of multiple zero modes that were not covered in the main section. Further we discuss how our results are related to a rigorous upper bound on the growth of the number of zero modes for strong magnetic fields, and we provide some interpretations for the fixed, universal background field that we had to introduce.

# 2 Two-dimensional case

First of all, we want to briefly recall the situation in two dimensions, because there will be some analogies with the class of three-dimensional zero modes that we shall discuss below. The two-dimensional Dirac equation is

$$\gamma_\mu(-i\partial_\mu - A_\mu(x))\Psi(x) = 0, \tag{1}$$

where $x = (x_1, x_2)$, $\mu = 1, 2$, $\gamma_\mu = \sigma_\mu$ and $\Psi$ is a two-component spinor. Both in Euclidean space $\mathbf{R}^2$ and on the two-sphere $S^2$ all zero modes are either left-handed (i.e., the lower component of $\Psi$ is zero) or right-handed (the upper component of $\Psi$ is zero). Further, a solution of the first type (left-handed) may be mapped into a solution of the second type by the simple replacement $A_\mu \to -A_\mu$, therefore we may restrict to the left-handed case

$$-i\begin{pmatrix} 0 & \partial_z - iA_z \\ \partial_{\bar z} - iA_{\bar z} & 0 \end{pmatrix}\rho^{1/2}(z,\bar z)e^{i\lambda(z,\bar z)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{2}$$

where

$$z = x_1 + ix_2\,, \quad \partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \tag{3}$$

$$A_z = \frac{1}{2}(A_1 - iA_2). \tag{4}$$

Here $\rho^{1/2}(z,\bar z)$ is a real, nonsingular function and $\lambda$ is a pure gauge factor that has to be determined accordingly (see below).

At this point we want to make some observations. Firstly, obviously only the left lower component $\partial_{\bar z} - iA_{\bar z}$ of the Dirac operator acts on the spinor in (2). Therefore, a spinor that solves (2) may be multiplied by an arbitrary holomorphic function $f(z)$ and still solves the same Dirac equation (2). A more complicated way of stating the same observation (which will be useful for the three-dimensional case) is as follows. We search for a function $f(z,\bar z)$ such that

$$-i\gamma_\mu(\partial_\mu f)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \tag{5}$$

then $f\Psi$ will formally solve the Dirac equation for the same Dirac operator (i.e., the same gauge potential) as $\Psi$. A possible choice for $f$ is $f = z$ and, as a consequence of the Leibnitz rule, arbitrary functions $f(z)$ of $z$ only are allowed. Observe that (5) implies

$$\det(-i\gamma_\mu \partial_\mu f) = (f_{,1})^2 + (f_{,2})^2 = 0, \tag{6}$$

which requires a complex $f$.

Secondly, from eq. (2) $A_\mu$ may be expressed in terms of $\rho$ and $\lambda$ as ($\epsilon_{12} = 1$)

$$A_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu \ln \rho + \lambda_{,\mu} \tag{7}$$

$$A_z = \frac{i}{2}\partial_z \ln \rho + \lambda_{,z}. \tag{8}$$

Now assume that $\rho(z, \bar{z})$ has a zero at some point $z_0$. As $\rho$ is real, let us assume that the zero is of the type $((z - z_0)(\bar{z} - \bar{z}_0))^\alpha$ for some $\alpha > 0$. This zero induces a singular contribution $A_\mu^{\rm sing}$ to the gauge potential $A_\mu$ (here $z_0 =: y_1 + iy_2$), where

$$A_\mu^{\rm sing} = \alpha\epsilon_{\mu\nu}\frac{x_\nu - y_\nu}{(\vec{x} - \vec{y})^2} = \alpha\partial_\mu \arg(z - z_0). \tag{9}$$

From the r.h.s. of (9) it is obvious that $A_\mu^{\rm sing}$ is in fact a pure gauge. Therefore, all singularities of $A_\mu$ due to zeros of $\rho$ of the above type may be gauged away by choosing the appropriate gauge functions

$$\lambda = -\alpha \arg(z - z_0) := \alpha \arctan \frac{x_2 - y_2}{x_1 - y_1} \tag{10}$$

in (2). As we want nonsingular gauge potentials, this gauge choice will be assumed in the sequel. However, $\lambda$ in (10) is not a single-valued function, and the gauge factor $\exp(i\lambda)$ of the spinor in (2) will be single-valued only provided that $\alpha = n \in \mathbf{N}$, i.e., only zeros of the above type of integer order are allowed for $\rho$. Zeros of other types (as e.g. $\rho = z + \bar{z}$, which is zero at $z = 0$) lead to singularities in $A_\mu$ that are not pure gauge, i.e., they lead to singular magnetic fields (14). They are, therefore, forbidden.

Now suppose that a zero mode for a non-singular gauge field is given and $\rho$ has some zeros of integer order of the allowed type as just described. For each zero $((z-z_0)(\bar{z}-\bar{z}_0))^n$ we may multiply the zero mode in (2) by the function $f(z) = (z-z_0)^{-n}$. This is a function of $z$ only, therefore the new spinor $f\Psi$ is a zero mode of the same Dirac operator. As a consequence, for each Dirac operator that admits zero modes there exists a zero mode (2) such that $\rho$ is nonzero everywhere, $\rho^{1/2}$ is strictly positive, $\rho^{1/2} > 0$. Further, the corresponding pure gauge terms (10) are absent, and we may assume that the gauge factor in (2) is absent altogether, $\lambda \equiv 0$, which corresponds to Lorentz gauge $\partial_\mu A_\mu = 0$ for the gauge potential in (7).

Let us assume that a spinor (2) is given with $\lambda = 0$, $\rho^{1/2} > 0$ and

$$\lim_{|z|\to\infty} \rho \sim (z\bar{z})^{-\alpha_\infty}. \tag{11}$$

Square-integrability of $\Psi$ in $\mathbf{R}^2$ implies $\alpha_\infty > 1$. If $n+1 > \alpha_\infty > n$ , $n \in \mathbf{N}$, then further square-integrable zero modes of the same Dirac operator may be constructed as

$$\Psi_k = z^k \rho^{1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad k = 0 \ldots n-1 \tag{12}$$

and the above-mentioned zero modes with some zeros of integer order may be recovered as linear combinations of (12). In addition, $\alpha_\infty$ determines the magnetic flux $\Phi$,

$$\Phi = \int d^2 x B = 2\pi \alpha_\infty \tag{13}$$

$$B = \partial_1 A_2 - \partial_2 A_1 = 2\partial_z \partial_{\bar{z}} \ln \rho. \tag{14}$$

Here we may follow two different approaches. Either we assume that our results really exist in Euclidean space. Then there are no further restrictions on $\alpha_\infty$. Further, whenever $\alpha_\infty = n \in \mathbf{N}$, then there are only $n-1$ square-integrable zero modes (12), because the one with $k = n-1$ is not square-integrable (its $L^2$ norm behaves as $\ln V$, where $V$ is the volume of space).

Or, on the other hand, we could interpret $z$ as a stereographic coordinate on the Riemann sphere. Then $z = \infty$ is a single point, and $\rho$ in (11) has a zero at this point if $\alpha_\infty \neq 0$, which leads to a singularity in $A_\mu$. This singularity, however, cannot be removed by a gauge transformation without introducing a singularity somewhere else. Instead, two different gauge potentials have to be chosen on different coordinate patches (e.g. on the northern and southern hemisphere), such that the difference of the two gauge potentials in the overlap region is a pure gauge $\partial_\mu \lambda$. The gauge function $\exp(i\lambda)$ (which acts on the zero mode) is single-valued only if $\alpha_\infty \equiv n_\infty \in \mathbf{N}$ and, consequently, the magnetic flux $\Phi = 2\pi n_\infty$ is quantised (in fact, this is just the well-known topology of the Dirac monopole). In addition, the zero modes are now normalised w.r.t. the integration measure on the sphere, therefore there are $n_\infty = \Phi/(2\pi)$ normalisable zero modes, in accordance with the index theorem.

# 3 Maps $S^2 \to S^2$ and Hopf maps $S^3 \to S^2$

The second homotopy group of the two-sphere is nontrivial, $\Pi_2(S^2) = \mathbf{Z}$, therefore maps $S^2 \to S^2$ are characterised by the integer winding number $w$. One way of describing them is by interpreting both $S^2$ as Riemann spheres and by introducing stereographic coordinates $z \in \mathbf{C}$ on both of them. A specific class of such maps $S^2 \to S^2$ may then be described by rational maps

$$R : \, z \to R(z) = \frac{P(z)}{Q(z)} \tag{15}$$

where $P(z)$ and $Q(z)$ are polynomials, and $z$ and $R(z)$ are interpreted as stereographic coordinates on the domain and target $S^2$, respectively. The winding number $w$ of this map is given by the degree of the map,

$$w = \deg(R) = \max(p, q) \tag{16}$$

where $p$ and $q$ are the degrees of the polynomials $P(z)$ and $Q(z)$ [15, 16]. Another possibility of computing the same winding number involves the pullback under $R(z)$ of the standard area two-form $\Omega$ on $S^2$ (in stereographic coordinates),

$$\Omega = \frac{2}{i}\frac{d\bar{z}dz}{(1+z\bar{z})^2}\, , \quad \int \Omega = 4\pi. \tag{17}$$

The pullback is ($'$ means derivative w.r.t. the argument)

$$R^*\Omega = \frac{2}{i}\frac{|R'(z)|^2}{(1+R\bar{R})^2}d\bar{z}dz \tag{18}$$

and obeys

$$\int R^*\Omega = 4\pi w \tag{19}$$

where $w$ is again the winding number (16).

However, rational maps are not the only types of functions that generate maps $S^2 \to S^2$. Instead of $R(z)$ we may e.g. choose the functions (here $z = u^{1/2}\exp(i\varphi)$, $u := z\bar{z}$ and $f$ is an at the moment arbitrary real function)

$$G(z,\bar{z}) = f(u)z^n =: g^{1/2}(u)e^{in\varphi} \tag{20}$$

which we shall need later on. The pullback of $\Omega$ under $G$ is ($' \equiv \partial_u$)

$$G^*\Omega = 2n\frac{g'}{(1+g)^2}dud\varphi \tag{21}$$

and its integral is

$$\int G^*\Omega = 4\pi n \int_0^\infty du \frac{g'}{(1+g)^2} = -4\pi n \frac{1}{1+g(u)}|_0^\infty . \tag{22}$$

If $g(0) = 0$ and $g(\infty) = \infty$, as holds e.g. for the rational maps $R(z) = z^n$, then the function $G$ in (20) defines a map $S^2 \to S^2$ with winding number $n$,

$$\int G^*\Omega = 4\pi n. \tag{23}$$

Apart from $g(u) \geq 0$, which follows from the definition of $g$, $g$ is not very much restricted in the intermediate range $0 < u < \infty$. Let us, e.g., assume that $g$ has a singularity at $u_1$ and a zero at $u_2$ (we assume $u_1 < u_2$ for this example), then the region $u \in [0, u_1]$ of the domain $S^2$ is mapped onto the target $S^2$ with winding number $+n$, the region $[u_1, u_2]$ is mapped onto the target $S^2$ with winding number $-n$, and the region $[u_2, \infty]$ is mapped onto the target $S^2$ with winding number $+n$, again. Therefore the net winding number is $n$.

Observe that it is possible to relate the pullback $R^*\Omega$ or $G^*\Omega$ to a magnetic field $B$ via (e.g. for $G$)

$$G^*\Omega =: Bdx_1dx_2 \equiv F, \tag{24}$$

where $F = (1/2)F_{\mu\nu}dx_\mu dx_\nu$ is the magnetic field strength two-form. However, all $B$'s that are constructed in this way have an even integer multiple of $2\pi$ as magnetic flux, $\Phi = \int d^2xB = 4\pi n = 2\pi \cdot 2n$, as is obvious from (23). Differently stated, if we want to formally express magnetic fields with magnetic fluxes that are odd integer multiples of $2\pi$ by maps $R$ or $G$, then we have to allow for square-root type, double-valued maps $R \sim z^{n/2}$ or $G \sim z^{n/2}$. This we shall need later on.

Hopf maps are maps $S^3 \to S^2$. The third homptopy group of the two-sphere is non-trivial as well, $\Pi_3(S^2) = \mathbf{Z}$, therefore such maps are characterised by an integer topological index, the so-called Hopf index. Hopf maps may be expressed, e.g., by maps $\chi : \mathbf{R}^3 \to \mathbf{C}$ provided that the complex function $\chi$ obeys $\lim_{|\vec{x}|\to\infty} \chi(\vec{x}) = \chi_0 = \text{const}$, where $\vec{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. The pre-images in $\mathbf{R}^3$ of points of the target $S^2$ (i.e., the pre-images of points $\chi = \text{const}$) are closed curves in $\mathbf{R}^3$ (circles in the related domain $S^3$). Any two different circles are linked $N$ times, where $N$ is the Hopf index of the given Hopf map $\chi$. Further, a magnetic field $\vec{\mathcal{B}}$ (the Hopf curvature) is related to the Hopf map $\chi$ via

$$\vec{\mathcal{B}} = \frac{2}{i}\frac{(\vec{\partial}\bar{\chi}) \times (\vec{\partial}\chi)}{(1+\bar{\chi}\chi)^2} = 2\frac{(\vec{\partial}T) \times \vec{\partial}\sigma}{(1+T)^2} \tag{25}$$

where $\chi = Se^{i\sigma}$ is expressed in terms of its modulus $S =: T^{1/2}$ and phase $\sigma$ at the r.h.s. of (25).

Mathematically, the curvature $\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij}dx_i dx_j$, $\mathcal{F}_{ij} = \epsilon_{ijk}\mathcal{B}_k$, is the pullback under the Hopf map, $\mathcal{F} = \chi^*\Omega$, of the standard area two-form $\Omega$ , (17), on the target $S^2$. Geometrically, $\vec{\mathcal{B}}$ is tangent to the closed curves $\chi = \text{const}$ (see e.g. [17, 18, 19, 20]; the authors of [20] describe Hopf curvatures slightly differently, by the Abelian projection of an $SU(2)$ pure gauge connection, which has some technical advantages). The Hopf index $N$ of $\chi$ may be computed from $\vec{\mathcal{B}}$ via

$$N = \frac{1}{16\pi^2}\int d^3x \vec{\mathcal{A}}\vec{\mathcal{B}} \tag{26}$$

where $\vec{\mathcal{B}} = \vec{\partial} \times \vec{\mathcal{A}}$.

Once a Hopf map $\chi$ is given, we may construct further Hopf maps by composing the Hopf map $\chi$ with maps $S^2 \to S^2$,

$$\chi_G : S^3 \xrightarrow{\chi} S^2 \xrightarrow{G} S^2 \tag{27}$$

where $G$ might be e.g. a $G(\chi, \bar{\chi})$ as in (20) or a rational map $R(\chi)$ as in (15). Further, if $\chi$ has Hopf index $N = 1$ and $G$ has degree (i.e. winding number) $n$, then the composed Hopf map $\chi_G$ has Hopf index $N = n^2$.

The simplest (standard) Hopf map $\chi$ with Hopf index $N = 1$ is

$$\chi = \frac{2(x_1 + ix_2)}{2x_3 - i(1 - r^2)} \tag{28}$$

with modulus and phase

$$T := \bar{\chi}\chi = \frac{4(r^2 - x_3^2)}{4x_3^2 + (1 - r^2)^2}\,, \quad \sigma = \arctan\frac{x_2}{x_1} + \arctan\frac{1 - r^2}{2x_3} \tag{29}$$

$$\sigma = \sigma^{(1)} + \sigma^{(2)}\,, \quad \sigma^{(1)} = \arctan\frac{x_2}{x_1}\,, \quad \sigma^{(2)} = \arctan\frac{1-r^2}{2x_3}. \tag{30}$$

Here the phase $\sigma$ is a sum of two terms $\sigma^{(1)}$ and $\sigma^{(2)}$, where $\sigma^{(1)}$ is multiply valued around the singular point $\chi = 0$ in target space, i.e., along the $x_3$ axis in the domain $\mathbf{R}^3$, and $\sigma^{(2)}$ is multiply valued around the singular point $\chi = \infty$, i.e., around the circle $\{\vec{x} \in \mathbf{R}^3 \setminus x_3 = 0, x_1^2 + x_2^2 = 1\}$. As $\chi$ in three dimensions will play a role similar to $z = x_1 + ix_2$ in two dimensions in Section 2, it is important to note a crucial difference in this respect. The same phase $\varphi = \arg z = \arctan(x_2/x_1)$ is multiply valued around both singular points $z = 0$ and $z = \infty$ in the two-dimensional case.

The simplest standard Hopf map, (28), leads to the Hopf curvature

$$\vec{\mathcal{B}} = \frac{16}{(1+r^2)^2}\vec{N} \tag{31}$$

and we have introduced the unit vector

$$\vec{N} = \frac{1}{1+r^2}\left(\begin{array}{c} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{array}\right) \tag{32}$$

($\vec{N}^2 = 1$) for later convenience.

# 4 Three-dimensional case

Here we want to study multiple solutions of the three-dimensional, Abelian Dirac equation (the Pauli equation)

$$-i\sigma_i\partial_i\Psi(x) = A_i(x)\sigma_i\Psi(x), \tag{33}$$

where $\vec{x} = (x_1, x_2, x_3)^{\mathrm{T}}$, $i, j, k = 1 \ldots 3$, $\Psi$ is a two-component, square-integrable spinor on $\mathbf{R}^3$, $\sigma_i$ are the Pauli matrices and $A_i$ is an Abelian gauge potential. Before starting the actual computations we want to mention some general aspects of the Dirac equation (33). Firstly, for any pair $(\Psi, \vec{A})$ that solves the Dirac equation (33), the zero mode $\Psi$ has to obey

$$\vec{\partial}\vec{\Sigma} = 0 \tag{34}$$

where $\vec{\Sigma}$ is the spin density of $\Psi$,

$$\vec{\Sigma} = \Psi^\dagger\vec{\sigma}\Psi. \tag{35}$$

Secondly, when a spinor $\Psi$ is given that obeys (34) (i.e., it is a possible zero mode), then the corresponding gauge potential $\vec{A}$ that solves the Dirac equation (33) together with $\Psi$ may actually be expressed in terms of $\Psi$ [4],

$$A_i = \frac{1}{|\vec{\Sigma}|}(\frac{1}{2}\epsilon_{ijk}\partial_j\Sigma_k + \mathrm{Im}\,\Psi^\dagger\partial_i\Psi)$$

$$= \frac{1}{2}\epsilon_{ijk}(\partial_j \ln|\vec{\Sigma}|)\mathcal{N}_k + \frac{1}{2}\epsilon_{ijk}\partial_j\mathcal{N}_k + \mathrm{Im}\,\widehat{\Psi}^\dagger\partial_i\widehat{\Psi} \tag{36}$$

where we have expressed $A_i$ in terms of the general unit vector and unit spinor

$$\vec{\mathcal{N}} = \frac{\vec{\Sigma}}{|\vec{\Sigma}|}, \quad \widehat{\Psi} = \frac{\Psi}{|\Psi^\dagger\Psi|^{1/2}} \tag{37}$$

for later convenience.

Next we want to discuss the simplest example of a zero mode that was already found in [4], because we need it as a starting point. The authors of [4] observed that a solution to this equation could be obtained from a solution to the simpler equation

$$-i\vec{\sigma}\vec{\partial}\Psi = h\Psi \tag{38}$$

for some scalar function $h(x)$. In this case the corresponding gauge field that obeys the Dirac equation (33) together with the spinor (38) is given by

$$A_i = h\frac{\Psi^\dagger\sigma_i\Psi}{\Psi^\dagger\Psi}. \tag{39}$$

In addition, they gave the following explicit example

$$\Psi = \frac{4}{(1+r^2)^{\frac{3}{2}}}(\mathbf{1} + i\vec{x}\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{40}$$

$$\vec{\Sigma} = \Psi^\dagger\vec{\sigma}\Psi = \frac{16}{(1+r^2)^2}\vec{N} \tag{41}$$

where $\vec{N}$ is the specific unit vector defined in (32) and we chose the factor 4 in (40) for later convenience. The spinor (40) obeys

$$-i\vec{\sigma}\vec{\partial}\Psi = \frac{3}{1+r^2}\Psi \tag{42}$$

and is, therefore, a zero mode for the gauge field

$$\vec{A} = \frac{3}{1+r^2}\frac{\Psi^\dagger\vec{\sigma}\Psi}{\Psi^\dagger\Psi} = \frac{3}{1+r^2}\vec{N} \tag{43}$$

with magnetic field $\vec{B} = \vec{\partial} \times \vec{A}$

$$\vec{B} = \frac{12}{(1+r^2)^2}\vec{N} \tag{44}$$

($\vec{N}$ is the unit vector defined in (32)).

Now we want to repeat the argument of (5) and (6) of the two-dimensional case, i.e., we assume that a function $\chi$ exists such that

$$(-i\sigma_j\partial_j\chi)(\mathbf{1} + i\vec{x}\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{45}$$

Consequently, $\chi^n \Psi$, $n \in \mathbf{Z}$ (where $\Psi$ is the zero mode (40)), are additional formal zero modes for the same gauge field (43). Condition (45) implies

$$\det(-i\vec{\sigma}\vec{\partial}\chi) = \sum_{i=1}^{3} \chi_{,i}\chi_{,i} = 0, \tag{46}$$

therefore, $\chi$ necessarily must be complex. Indeed, such a function $\chi$ fulfilling (45) exists. It is just the simplest Hopf map $\chi$, (28), as may be checked easily. For the formal zero modes $\chi^n \Psi$ we observe the following two points. Firstly, $n$ has to be integer, because only integer powers of $\chi$ lead to a single-valued spinor $\chi^n \Psi$. Secondly, $\chi^n \Psi$ is singular for all $n \in \mathbf{Z} \setminus \{0\}$, because $\chi$ is singular along the circle $\{\vec{x} \in \mathbf{R}^3 \setminus x_3 = 0\,, x_1^2 + x_2^2 = 1\}$ and zero along the $x_3$ axis. Therefore, the formal zero modes $\chi^n \Psi$, with $\Psi$ given in (40), are not acceptable. However, we shall find some zero modes, different from (40), where multiplication with $\chi^n$ will lead to acceptable new zero modes for some $n \neq 0$.

For this purpose, let us observe that the spin density (41) of the simplest zero mode (40) is in fact equal to the Hopf curvature (31) of the simplest Hopf map (28) (we chose the constant factor 4 in (40) in order to achieve this equality; otherwise $\vec{\Sigma}$ would only be proportional to the Hopf curvature, which is enough for our purposes). As a consequence (see eq. (25))

$$\vec{\Sigma}^{(M)} := e^{M(\chi,\bar{\chi})}\vec{\Sigma} = \frac{16}{(1+r^2)^2} e^{M(\chi,\bar{\chi})}\vec{N} \tag{47}$$

still is the spin density of a zero mode, i.e., it still obeys $\vec{\partial}\vec{\Sigma}^{(M)} = 0$. Here $M(\chi, \bar{\chi})$ is a real function of $\chi$ and $\bar{\chi}$. The corresponding zero mode reads

$$\Psi^{(M)} = e^{i\Lambda} e^{M/2} \Psi = e^{i\Lambda} e^{M/2} \frac{\mathbf{1} + i\vec{\sigma}\vec{x}}{(1+r^2)^{3/2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{48}$$

where $\Lambda$ is a gauge function that has to be determined accordingly (analogously to our discussion in Section 2; see below). $\Psi^{(M)}$ is proportional to the simplest zero mode (40), therefore it remains true that additional formal zero modes for the same Dirac operator may be constructed from $\Psi^{(M)}$ by multiplication with powers $\chi^n$ of $\chi$, (28).

At this point we want to present some first examples of such multiple zero modes (these examples were already discussed in [6]). For this purpose, we need some more results of [4]. The authors of [4] observed that, in addition to their simplest solution (40), they could find similar solutions to eq. (38) with higher angular momentum. Using instead of the constant spinor $(1,0)^{\mathrm{T}}$ the spinor

$$\Phi_{l,m} = \begin{pmatrix} \sqrt{l+m+1/2}\, Y_{l,m-1/2} \\ -\sqrt{l-m+1/2}\, Y_{l,m+1/2} \end{pmatrix} \tag{49}$$

(where $m \in [-l-1/2\,,\, l+1/2]$ and $Y$ are spherical harmonics), they found the solutions

$$\Psi_{l,m} = r^l (1+r^2)^{-l-\frac{3}{2}} (\mathbf{1} + i\vec{x}\vec{\sigma}) \Phi_{l,m} \tag{50}$$

$$\vec{A}_{l,m} = (2l+3)(1+r^2)^{-1}\frac{\Psi^\dagger_{l,m}\vec{\sigma}\Psi_{l,m}}{\Psi^\dagger_{l,m}\Psi_{l,m}}. \tag{51}$$

Specifically, for maximal magnetic quantum number $m = l + 1/2$, these solutions read

$$\Psi_l := \Psi_{l,l+1/2} = \frac{Y_{l,l}r^l}{(1+r^2)^{l+3/2}}(\mathbf{1} + i\vec{x}\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix} \tag{52}$$

$$\vec{A}^{(l)} = \frac{3+2l}{1+r^2}\vec{N} \tag{53}$$

$$\vec{B}^{(l)} = \frac{4(3+2l)}{(1+r^2)^2}\vec{N} \tag{54}$$

(where we have omitted an irrelevant constant factor in (52)). Hence, $\Psi_l$ is proportional to the simplest zero mode (40) and is, therefore, still an eigenvector of the matrix $-i\sigma_j\partial_j\chi$ with eigenvalue zero. Further, the zero mode $\Psi_l$ may be rewritten as (again, we ignore irrelevant constant factors)

$$\Psi_l = e^{il\varphi}\Big(\frac{T}{1+T}\Big)^{l/2}(1+r^2)^{-3/2}(\mathbf{1} + i\vec{x}\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix} \tag{55}$$

where we introduced polar coordinates $(x_1, x_2, x_3) \to (r, \theta, \varphi)$, $T$ is the squared modulus (29), and (up to an irrelevant constant)

$$Y_{l,l} = e^{il\varphi}\sin^l\theta = e^{il\varphi}\frac{(r^2 - x_3^2)^{l/2}}{r^l} = e^{il\varphi}\frac{(1+r^2)^l}{r^l}\Big(\frac{T}{1+T}\Big)^{l/2}. \tag{56}$$

Taking further into account that $\varphi = \arctan(x_2/x_1) = \sigma^{(1)}$ we conclude that the spinors

$$\Psi_{n,l} = \chi^{-n}\Psi_l = e^{i(l-n)\sigma^{(1)} - in\sigma^{(2)}}\frac{T^{(l-n)/2}}{(1+T)^{l/2}}\Psi\,, \quad n = 0, \dots l \tag{57}$$

are non-singular, square-integrable zero modes for the same gauge field $\vec{A}^{(l)}$ and, therefore, the Dirac operator with gauge field $\vec{A}^{(l)}$ given by (53) has $l+1$ square-integrable zero modes (57). Here $\sigma^{(1)}$ and $\sigma^{(2)}$ are the two terms (30) of the phase of the simplest Hopf map (28).

At this point several remarks are necessary. Firstly, observe that the function $\exp(M)$, as defined in (47), for the zero modes (57) reads

$$e^M = \frac{T^{l-n}}{(1+T)^l} \tag{58}$$

$$\lim_{T\to 0} e^M \sim T^{l-n}\,, \quad \lim_{T\to\infty} e^M \sim T^{-n} \tag{59}$$

Hence, $\exp(M)$ has a zero of order $l-n$ at $T=0$ and a zero of order $n$ at $T=\infty$. As in the two-dimensional case, these zeros introduce singularities in the gauge potentials,

which are cured by the pure gauge functions $(l-n)\sigma^{(1)}$ and $-n\sigma^{(2)}$, respectively, leading to the well-behaving gauge potentials (53). In contrast to the two-dimensional case, the singularity at $\chi = \infty$ may be cured independently, i.e., without introducing singularities somewhere else (for an explanation see below).

Secondly, we observe that already the simplest magnetic field (44) (for $l = 0$) is proportional but not equal to the Hopf curvature (31) (the magnetic field has a factor of 12 instead of 16, i.e., they differ by $4(1+r^2)^2\vec{N}$). Here we will take the following point of view. We assume that this difference is related to a fixed, universal background magnetic field $\vec{B}^{\rm b}$,

$$\vec{B}^{\rm b} = -\frac{4}{(1+r^2)^2}\vec{N} \tag{60}$$

which couples to the Fermion via the Dirac operator but is "non-dynamical" otherwise. Then for the "dynamical" part $\widetilde{B}_j := B_j - B_j^{\rm b}$ of $B_j$ it holds that

$$\widetilde{B}_j = B_j - B_j^{\rm b} = \frac{16}{(1+r^2)^2}N_j = \mathcal{B}_j \tag{61}$$

where $\mathcal{B}_j$ is the Hopf curvature (31). We immediately find that this feature continues to hold for all the higher $B_j^{(l)}$ in (54),

$$\widetilde{B}_j^{(l)} = B_j^{(l)} - B_j^{\rm b} = \frac{16(1+l/2)}{(1+r^2)^2}N_j. \tag{62}$$

These $\widetilde{B}_j^{(l)}$ are Hopf curvatures for the Hopf maps

$$\chi^{(l)} = T^{1/2}e^{i(1+l/2)\sigma} \tag{63}$$

where $T$ and $\sigma$ are given in (29). We find that we have to allow for double-valued, square-root type Hopf maps if we want to relate all $\widetilde{B}_j^{(l)}$ to Hopf curvatures. Further, we find the relation

$$N = \Big(\frac{k+1}{2}\Big)^2 \tag{64}$$

between the Hopf index $N$ and the number $k = l+1$ of zero modes. We will find that after the subtraction of the universal background field (60) all these features continue to hold for a much wider class of solutions to the Dirac equation.

In order to discuss this wider class, let us go back to the general zero mode (48) which depends on a function $M(\chi,\bar{\chi})$ and a pure gauge function $\Lambda$. The corresponding gauge potential $\vec{A}^{(M)}$ that obeys the Dirac equation together with $\Psi^{(M)}$ may be computed from (36),

$$A_j^{(M)} = A_j + \frac{1}{2}\epsilon_{jkl}M_{,k}N_l + \Lambda_{,j} \tag{65}$$

$$= A_j + \frac{i}{2}(M_{,\chi}\chi_{,j} - M_{,\bar{\chi}}\bar{\chi}_{,j}) + \Lambda_{,j} \tag{66}$$

where the second line follows after some algebra. Here $A_j$ is the gauge potential (43) of the simplest zero mode (40) and $N_l$ is the unit vector (32).

At this point we have to discuss the possible singularities of $\vec{A}^{(M)}$, which will determine $\Lambda$ and, at the same time, pose some restrictions on $\exp(M)$, as in the two-dimensional case (Section 2). As in the two-dimensional case, zeros of $\exp(M)$ cause singularities of $\vec{A}^{(M)}$, and in order to cause only removable singularities, these zeros have to be of the type (with possible multiplicity $n$)

$$((\chi - z_i)(\bar{\chi} - \bar{z}_i))^n =: \zeta^n \bar{\zeta}^n \tag{67}$$

which implies for the above expression (66) (without the pure gauge piece $\Lambda_{,j}$)

$$\frac{i}{2}(M_{,\chi}\chi_{,l} - M_{,\bar{\chi}}\bar{\chi}_{,l}) \sim \frac{in}{2}\frac{\bar{\zeta}\chi_{,l} - \zeta\bar{\chi}_{,l}}{\zeta\bar{\zeta}} + \ \ldots \tag{68}$$

where the remainder is regular at $\zeta = 0$. The above singularity may be compensated by the pure gauge factor

$$\Lambda = -n \ \arctan \frac{i(\zeta - \bar{\zeta})}{\zeta + \bar{\zeta}}. \tag{69}$$

Indeed ($\zeta_{,l} \equiv \chi_{,l}$),

$$\Lambda_{,l} = -\frac{in}{2}\frac{\bar{\zeta}\zeta_{,l} - \zeta\bar{\zeta}_{,l}}{\zeta\bar{\zeta}} \tag{70}$$

precisely cancels the singular term (68). The spinor in (48) is multiplied by the gauge factor $\exp(i\Lambda)$. This factor is single-valued only if the order $n$ of the zero is integer, because $\Lambda$ in (69) is a multiply-valued function.

In fact, this is not yet the whole story about singularities in $\bar{A}_l^{(M)}$. The point is that the expression

$$\frac{i}{2}\frac{\bar{\chi}\chi_{,l} - \chi\bar{\chi}_{,l}}{\chi\bar{\chi}} \tag{71}$$

is singular in the limit $\chi \to \infty$ as well, as may be easily checked. Further, the derivatives of the gauge factors, (70), for all the zeros (67) produce this expression (71) for $\chi \to \infty$, because $\lim_{\chi\to\infty} \zeta = \chi$. In addition, $\exp(M)$ may cause a similar term (71) for $\vec{A}^{(M)}$ if it behaves as

$$\lim_{|\chi|\to\infty} \exp(M) \sim |\chi\bar{\chi}|^{-n_\infty} \equiv T^{-n_\infty}. \tag{72}$$

Here $n_\infty$ has to be a positive integer or zero, as we shall see immediately. Further, $\exp(M)$ has to reach the limiting value sufficiently fast,

$$\lim_{|\chi|\to\infty} (T^{n_\infty} \exp(M)) \sim 1 + cT^{-\alpha}\,, \quad \alpha \geq 1 \tag{73}$$

($c$ is some constant) as will be explained below. Therefore, altogether we have to compensate

$$(-n_\infty + \sum_i n_i)\frac{i}{2}\frac{\bar{\chi}\chi_{,l} - \chi\bar{\chi}_{,l}}{\chi\bar{\chi}} \tag{74}$$

by an additional gauge transformation, without introducing further singularities at $\chi = 0$ (here $n_i$ are the multiplicities of the zeros $z_i$ of $\exp(M)$).

Fortunately this is possible for the following reason. If we were to compensate (74) by the full gauge function

$$\Lambda = (n_\infty - \sum_i n_i) \arctan \frac{i(\chi - \bar{\chi})}{\chi + \bar{\chi}} = -(n_\infty - \sum_i n_i)\sigma \tag{75}$$

(where $\sigma$ is the phase of $\chi$ given in (29)), this would introduce a singularity at $\chi = 0$. However, $\sigma$ is the sum of two terms $\sigma = \sigma^{(1)} + \sigma^{(2)}$ (see (30)), where $\sigma^{(1)}_{,l}$ is singular at $\chi = 0$ and $\sigma^{(2)}_{,l}$ is singular at $\chi = \infty$. Therefore, we may cancel the singularity of (74) without introducing further singularities by performing an additional gauge transformation using only $\sigma^{(2)}$,

$$\Lambda = -(n_\infty - \sum_i n_i)\sigma^{(2)}. \tag{76}$$

Obviously, $n_\infty$ has to be integer for (76) to be an acceptable gauge function.

We want to emphasise again here that there is a crucial difference to the two-dimensional case (Section 2, last paragraph), where no gauge choice was possible to achieve a non-singular gauge potential for all $z$. The reason for this difference lies in the different topological features of the underlying spaces $S^2$ and $S^3$, respectively. In fact, the second cohomology group of the $S^2$ is non-trivial, $H_2(S^2) = \mathbf{Z}$. Therefore, it is not possible to find a globally defined gauge potential on $S^2$ for magnetic fields with non-zero (quantised) magnetic flux. On the other hand, $H_2(S^3) = 0$, therefore it is always possible to find a well-behaving non-singular gauge potential for a well-behaving non-singular magnetic field.

One consequence of the above discussion is that (as in the two-dimensional case) the zeros $((\chi - z_0)(\bar{\chi} - \bar{z}_0))^n$ may be removed by multiplying the corresponding zero mode with the holomorphic function (in the variable $\chi$) $(\chi - z_0)^{-n}$ without changing the Dirac operator. Therefore, for each Dirac operator that admits zero modes there exists one zero mode such that $\exp(M/2)$ is strictly positive, $\exp(M/2) > 0$ for all $\chi < \infty$. This we will assume in the sequel. Further we assume

$$\lim_{|\chi| \to \infty} \exp(M) \sim T^{-n_\infty} \tag{77}$$

as in (72), (73). The corresponding zero mode is

$$\Psi^{(M)} = e^{i\Lambda} e^{M/2} \Psi \tag{78}$$

where $\Lambda$ is given in (76) (with $n_i = 0$). Additional non-singular, square-integrable zero modes for the same Dirac operator are

$$\Psi^{(M)}_n = \chi^n \Psi^{(M)}, \quad n = 0, \ldots n_\infty \tag{79}$$

i.e., there are $k = n_\infty + 1$ zero modes. As in the two-dimensional case, zero modes with arbitrary allowed zeros may be constructed as linear combinations of the above zero modes (79).

Finally we have to discuss the related magnetic field. The magnetic field $B_i^{(M)} = \epsilon_{ijk}\partial_j A_k^{(M)}$ corresponding to the gauge potential (65) is

$$B_l^{(M)} = B_l + \frac{1}{2}[M_{,\chi}(\chi_{,lk}N_k + \chi_{,l}N_{k,k} - \chi_{,kk}N_l - \chi_{,k}N_{l,k}) +$$
$$M_{,\bar{\chi}}(\bar{\chi}_{,lk}N_k + \bar{\chi}_{,l}N_{k,k} - \bar{\chi}_{,kk}N_l - \bar{\chi}_{,k}N_{l,k})$$
$$- (M_{,\chi\chi}\chi_{,k}\chi_{,k} + M_{,\bar{\chi}\bar{\chi}}\bar{\chi}_{,k}\bar{\chi}_{,k} + 2M_{,\chi\bar{\chi}}\chi_{,k}\bar{\chi}_{,k})N_l] \tag{80}$$

where $B_l$ is the magnetic field (44). After some tedious algebra we find that only the coefficient of $M_{,\chi\bar{\chi}}$ is nonzero, i.e.,

$$\chi_{,lk}N_k + \chi_{,l}N_{k,k} - \chi_{,kk}N_l - \chi_{,k}N_{l,k} = 0 \tag{81}$$

$$\chi_{,k}\chi_{,k} = 0 \tag{82}$$

$$\chi_{,k}\bar{\chi}_{,k} = 8\frac{(1+\chi\bar{\chi})^2}{(1+r^2)^2} \tag{83}$$

and, therefore

$$B_l^{(M)} = B_l - 8\frac{(1+\chi\bar{\chi})^2}{(1+r^2)^2}M_{,\chi\bar{\chi}}N_l. \tag{84}$$

Obviously, $\vec{B}^{(M)}$ will be finite in the limit $|\chi| \to \infty$ only if $\lim_{|\chi|\to\infty} M_{,\chi\bar{\chi}} \sim |\chi\bar{\chi}|^{-2-\epsilon}$, $\epsilon \geq 0$. This corresponds to eq. (73) and explains our remark that $\exp(M)$ has to reach its limiting value sufficiently fast.

As in (61), we now have to subtract the background magnetic field (60) in order to be able to relate the resulting "dynamical" magnetic field $\widetilde{B}_l^{(M)}$ to Hopf maps. We find

$$\widetilde{B}_l^{(M)} = \Big(1 - \frac{1}{2}(1+\chi\bar{\chi})^2 M_{,\chi\bar{\chi}}\Big)\mathcal{B}_l \tag{85}$$

where $\vec{\mathcal{B}}$ is the Hopf curvature (31).

At this point we want to specialise to the class of functions

$$M(\chi,\bar{\chi}) = M(\chi\bar{\chi}) \equiv M(T)\,, \quad M' \leq 0 \tag{86}$$

($' \equiv \partial_T$) because we want to relate them to Hopf maps of the type (27) where the function $G$ is given by (20). For these functions $M(T)$, (85) simplifies to

$$\widetilde{B}_l^{(M)} = \Big(1 - \frac{1}{2}(1+T)^2(M' + TM'')\Big)\mathcal{B}_l. \tag{87}$$

We want to re-express this magnetic field as a Hopf curvature $\vec{\mathcal{B}}^{(G)}$ for the Hopf map

$$\chi^{(G)} = g^{1/2}(T)e^{im\sigma} \tag{88}$$

which is a composition of the standard Hopf map (28) and a map $S^2 \to S^2$ of the type $G$ as in (20). The Hopf curvature $\vec{\mathcal{B}}^{(G)}$ is

$$\vec{\mathcal{B}}^{(G)} = 2m\frac{(\vec{\partial}g) \times \vec{\partial}\sigma}{(1+g)^2} = m\frac{g'(1+T)^2}{(1+g)^2}\vec{\mathcal{B}} \tag{89}$$

which is indeed a Hopf curvature if $g(0) = 0$, $g(\infty) = \infty$, see (22). Equality of (87) and (89) implies

$$-m\Big(\frac{1}{1+g}\Big)' = -\Big(\frac{1}{1+T}\Big)' - \frac{1}{2}(M'T)' \tag{90}$$

or upon integration

$$\frac{m}{1+g} = \frac{1}{1+T} + \frac{1}{2}TM' + \frac{1}{2}n_\infty \tag{91}$$

$$m = 1 + \frac{1}{2}n_\infty \tag{92}$$

(where we have chosen an appropriate constant of integration in (91)). Here $M' \leq 0$ (together with $\exp(M) > 0$ and condition (77)) is a sufficient condition to ensure $g \geq 0$.

Therefore, we find that for all the zero modes of the type (78), (86) the corresponding magnetic fields may indeed be expressed as Hopf curvatures, provided that we allow for double-valued Hopf maps, $m = 1 + (n_\infty/2)$, whenever the Dirac operator has an even number of zero modes. In addition, we confirm the general relation between Hopf index $N = m^2$ and the number of zero modes $k = n_\infty + 1$,

$$N = \Big(\frac{k+1}{2}\Big)^2. \tag{93}$$

# 5 Discussion

We have found a class of magnetic fields (87) that are the Hopf curvatures of the Hopf maps (88) (after the subtraction of the fixed background field (60)). The corresponding Dirac operator shows a degeneracy of zero modes, where the number of zero modes is related to the Hopf index via eq. (93). Here we had to allow for double-valued Hopf maps whenever the number of zero modes is even.

Further, we imposed some restrictions on the zero modes (i.e., on the functions $M(\chi, \bar{\chi})$) because we wanted to relate the corresponding magnetic fields to the specific, simple type (88) of Hopf maps. We think that these restrictions are a mere technicality, and that abandoning these restrictions will just lead to more complicated Hopf maps. One specific type of such Hopf maps, different from (88), is easily accessible and leads to results that are in complete agreement with the ones we have described above, therefore we want to describe it briefly.

Recall that there exists a class of Hopf maps that are a composition of the standard Hopf map with an arbitrary rational map $R(\chi) = P(\chi)/Q(\chi)$, see (15). The corresponding

Hopf curvature reads ($'\equiv$ derivative w.r.t. the argument)

$$\mathcal{B}_l^{(R)} = \frac{|P'Q - PQ'|^2}{(|P|^2+|Q|^2)^2}(1+\chi\bar{\chi})^2\mathcal{B}_l = \widetilde{B}_l^{(M_R)} \tag{94}$$

($P$ and $Q$ do not have a common zero), where we have already indicated on the r.h.s. of (94) that there exists a magnetic field $\widetilde{B}_l^{(M_R)}$ for some zero mode $\Psi^{(M_R)}$. In fact, $\exp(M_R)$ reads

$$\exp(M_R) = \frac{(1+\chi\bar{\chi})^2}{(|P|^2+|Q|^2)^2} \tag{95}$$

$$\lim_{|\chi|\to\infty} \exp(M_R) = (\chi\bar{\chi})^{-2(w-1)} \tag{96}$$

where $w$ is the degree (16) of the rational map $R$. Therefore there are $k = 2w-1$ zero modes

$$\Psi_n^{(M_R)} = \chi^n \Psi^{(M_R)}, \quad n = 0, \ldots 2(w-1). \tag{97}$$

In addition, the corresponding magnetic field $\widetilde{B}_l^{(M_R)}$ (after the subtraction of the background field) is indeed equal to the Hopf curvature (94), as may be computed easily with the help of eq. (84). The Hopf index is $N = w^2$, therefore the relation (93) between Hopf index and number of zero modes is confirmed once more.

This class of solutions has another interesting feature. A zero mode may be constructed (a specific linear combination of the zero modes (97)) such that its spin density $\Sigma_l$ equals the magnetic field $\widetilde{B}_l^{(M_R)}$. Hence in addition to the Dirac equation (33) this solution obeys the equation $\Sigma_l = \widetilde{B}_l$. This system of equations of motion is generated by the Lagrangian density

$$\mathcal{L} = \Psi^\dagger \sigma_j(-i\partial_j - A_j)\Psi + \frac{1}{2}\widetilde{A}_j\widetilde{B}_j, \tag{98}$$

where the background field is coupled to the Fermion, but it is absent in the second, "kinetic" term (the Abelian Chern–Simons term). This explains why we called $\widetilde{A}_l$ the "dynamical" gauge potential (for details see [21, 22], where these solutions ("Hopf instantons") were discussed in depth).

Another point that we want to mention here is the fact that our results may be used to estimate the number of zero energy bound states (zero modes) for strong magnetic fields. This is seen especially easily for the higher angular momentum zero modes (57), because the magnetic fields (54) for higher angular momentum $l$ are just multiples of the simplest magnetic field (44). Therefore, the number $k = l+1$ of zero modes for strong magnetic fields (i.e. large $l$) behaves like

$$k = l+1 \sim c\int d^3x |\vec{B}^{(l)}| \tag{99}$$

(it holds that $\lim_{|\vec{x}|\to\infty} |\vec{B}^{(l)}| \sim r^{-4}$, therefore the integral in (21) exists), i.e., $k$ grows linearly with the strength of the magnetic field (here $c$ is some constant). This remains true in a certain sense for our other solutions. From (93) we infer that the number of

zero modes $k$ behaves like $k \sim N^{1/2}$ for large $k$ ($N$ is the Hopf index). Further, as $N \sim \int d^3x \widetilde{A}_j \widetilde{B}_j$, the number of zero modes grows like $\lambda$ under a rescaling $\widetilde{A}_j \to \lambda \widetilde{A}_j$, $\widetilde{B}_j \to \lambda \widetilde{B}_j$. This is well within the rigorous upper bound on the possible growth of the number of zero modes

$$k \sim c' \int d^3x |\vec{B}|^{3/2} \tag{100}$$

that was first stated in [4] and later derived in [9] (here $c'$ is a constant; the difference between $\widetilde{B}_j$ and $B_j$ is unimportant for strong fields, because the background magnetic field (60) is the same for all magnetic fields). We should mention here that it is, in principle, possible that the Dirac operators of our magnetic fields have in fact more zero modes than we have discovered with our methods, which would imply that the true number of zero modes is closer to the rigorous upper bound (100).

Observe that it was possible to relate our magnetic fields to Hopf curvatures only after the subtraction of the fixed, universal background field (60) (although the existence and degeneracy of the zero modes per se does not require the background field). Further, the above-mentioned solutions to the equations of motion of the Chern–Simons and Fermion system (98) only exist in the presence of this background field, as well ([21, 22]). Therefore, this background field (60) seems to be rather fundamental for our discussion, and one wonders whether it admits some further interpretation. We cannot yet give a final answer to this question, but we want to mention two possible interpretations that were already given in [21]. On one hand, if one compares the background magnetic field (60) with the magnetic fields (54) of the higher angular momentum zero modes (52), then one realises that changing the angular momentum by one unit produces a change of the corresponding magnetic field that is precisely minus two times the background field (60). It is, therefore, tempting to conjecture that the background field is somehow related to the half-integer angular momentum (spin) of the Fermion. Of course, this is just an observation at this point, because a mechanism that generates this background field is still missing.

On the other hand, it is possible to re-interpret the background gauge potential $\vec{A}^{\mathrm{b}} = -(1+r^2)^{-1}\vec{N}$ of the background magnetic field (60) as a spin connection $\omega$ in the Dirac equation (33) on a conformally flat manifold with torsion. Generally, the Dirac operator with spin connection reads (see e.g. [23] for details)

$$\not{D} = \gamma^a E_a{}^\mu (\partial_\mu + A_\mu + \frac{1}{4}[\gamma_b, \gamma_c]\omega^{bc}{}_\mu) \tag{101}$$

where $\gamma^a$ ($\equiv \sigma^a$ in our case) are the usual Dirac matrices, $E_a{}^\mu$ is the inverse vielbein and $\omega^{bc}{}_\mu$ is the spin connection (here $\mu, \nu$ are Einstein (i.e., space time) indices and $a, b, c$ are Lorentz indices). Our Dirac equation (33) may be rewritten in the form of eq. (101) provided that the vielbein is conformally flat, $E_a{}^\mu = f\delta_a^\mu$, where $f$ is an arbitrary function. Using $[\sigma_b, \sigma_c] = 2i\epsilon_{bcd}\sigma^d$ we find

$$\frac{i}{2}\delta_a^k \epsilon_{bcd}\sigma^a\sigma^d\omega^{bc}{}_k \stackrel{!}{=} \delta_a^k \sigma^a A_k^{\mathrm{b}} \tag{102}$$

(here $k$ is an Einstein index in three dimensions). The l.h.s. of (102) has to be anti-symmmetric in $a, d$, i.e., the quantity $\widetilde{\omega}_{da} := \delta_a^k \epsilon_{bcd}\omega^{bc}{}_k$ obeys $\widetilde{\omega}_{da} = -\widetilde{\omega}_{ad}$. This leads

to $\widetilde{\omega}_{ab} = \epsilon_{abc}\delta^k_c A^{\mathrm{b}}_k$. If we further assume $\omega^{ab}{}_k = -\omega^{ba}{}_k$ (i.e., covariant constancy of the metric) then we find that

$$\omega_{abk} = \delta_{ka}A^{\mathrm{b}}_b - \delta_{kb}A^{\mathrm{b}}_a \tag{103}$$

(where $A^{\mathrm{b}}_a \equiv \delta^k_a A^{\mathrm{b}}_k$, i.e., it is *not* the Lorentz vector $E_a{}^k A^{\mathrm{b}}_k$). Finally, we find for the torsion $T$ (expressed in Lorentz indices only)

$$2T_{abc} = (\delta_{ab}\delta^k_c - \delta_{ac}\delta^k_b)\partial_k f - (\omega_{abc} - \omega_{acb}) \tag{104}$$

where

$$\omega_{abc} = E_c{}^k \omega_{abk} = f\delta^k_c \omega_{abk}. \tag{105}$$

Hence, with $\omega_{abk}$ given by (103), we may freely choose a conformally flat metric (i.e., conformal factor $f$) and compute the resulting torsion via (104). Due to the form of $\omega_{abk}$ (i.e., $\vec{A}^{\mathrm{b}}$) it is, however, not possible to choose a conformal factor such that the torsion is zero. On the other hand, it is possible to choose the flat metric $f = 1$, so that (the anti-symmetric part of) the spin connection is given just by the torsion.

Finally we want to point out that some important questions still remain to be answered.

Firstly, all our zero modes are of a specific type. They are multiples (by a scalar function) of the simplest spinor (40). There exist, of course, zero modes of a different type (see e.g. [5]). By the very methods of [5], only one zero mode per Dirac operator (i.e., per gauge potential) could be constructed. We believe that the methods of this paper may, in principle, be adapted to address the question of a degeneracy of zero modes for more general Dirac operators, like those in [5].

Secondly, all our magnetic fields are Hopf curvatures after the subtraction of the background magnetic field (60), where one has to allow for double-valued Hopf maps in the case of an even number of zero modes. This immediately leads to the question whether this feature can be proven in general, and whether the existence and degeneracy of zero modes may be explained on topological grounds, as is the case in even dimensions.

Thirdly, the topological interpretation of our magnetic fields (as Hopf curvatures) was possible only after the introduction of the universal background magnetic field (60). We already provided some possible interpretations of this background field, but we think that it plays a rather fundamental role in the whole problem and, therefore, deserves some further investigation.

Anyhow, we think that our results should be relevant for some future developments in mathematical physics, as well as for the understanding of non-perturbative aspects of quantum electrodynamics, especially in three dimensions.

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